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Quantum detector in an accelerated cavity

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Abstract. A quantum field and a detector are enclosed in a uniformly accelerated cavity. *The field is in its ground state. If the detector is accelerated together with the cavity,* it will *not* be excited by the vacuum fluctuations of the field. On the other hand, an *inertial* detector will be excited.

1. Vacuum fluctuations

In quantum field theory, the 'vacuum' is defined as the lowest energy state of a *field*—that is a dynamical system with an infinite number of degrees of freedom. A free field with linear equations of motion can be resolved into normal modes, such as standing waves. Each mode has a fixed frequency, ω, and behaves as a harmonic oscillator. Quantum theory then predicts that the vacuum fluctuations (namely, the zero point motion of all these harmonic oscillators) can excite a suitable detector from a lower to a higher energy level. For example, a detector moving with a constant acceleration g in a Minkowski vacuum reacts as if it were in a thermal bath (Davies 1975, Unruh 1976). This is called the Unruh effect. It is related to the fluctuation–dissipation theorem (Candelas and Sciama 1977) and results from the autocorrelation of the field variables along the world line of the detector. The vacuum fluctuations appear to have a Planckian spectrum, with a temperature $kT = g\hbar/2\pi c$.

For any reasonable linear acceleration, this temperature is exceedingly low. The issue which motivated this article is whether the Unruh effect is observable, *as a matter of principle*. Leaving aside any mundane technological considerations, we must at least verify that we have a 'true' vacuum, rather than black-body radiation due to the cosmic background or to other sources. The detector must therefore be *shielded* from parasitic sources, and moreover we must *cool* the walls of the enclosure where the experiment is performed, to well below the Unruh temperature.

This, however, creates a radically new situation, because the presence of boundaries affects the dynamical properties of a quantum field by altering the frequencies of its normal modes. Finite-size effects have been known for a long time, both theoretically (Casimir 1948) and experimentally (Spaarnay 1958). A recent discussion by Gerlach (1989) follows an approach similar to ours. In the present problem, boundary effects cannot be neglected if the field is restricted to a domain smaller than the wavelength corresponding to the Unruh temperature, which is about c^2/g (one light-year for the Earth's g).

To gain a better understanding of the respective roles played by the detector and by the radiation modes, we consider here a closely related problem, that of a quantum field in a uniformly accelerated cavity. It is convenient to use dimensionless Rindler coordinates θ and ξ, defined by

$$ct = Le^{\xi} \sinh\theta \qquad \text{and} \qquad x = Le^{\xi}\cosh\theta \tag{1}$$

where L is an arbitrary constant with the dimensions of a length. Any world line with constant ξ has (in the original Lorentz frame) a constant proper acceleration $g = c^2/Le^{\xi}$, as can be seen from $x^2 = c^2t^2 + Le^{2\xi}$.

Note that the Rindler coordinates cover only the wedge $x > |ct|$. They are illustrated in figure 1, where ξ_1 and ξ_2 are the 'floor' and 'ceiling' of the cavity, respectively. (The y and z coordinates are ignored, for simplicity. However, most of the results presented here remain valid in a three-dimensional space.)

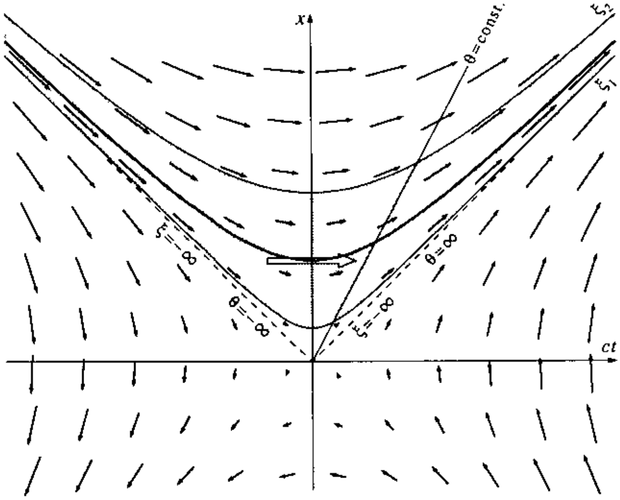

Figure 1. The shaded area is the world tube swept by an accelerated cavity, in Minkowski and Rindler coordinates (the latter cover only the wedge $x > c|t|$). The thick line between the floor ξ_1 and the ceiling ξ_2 is the world line of a detector at rest in the cavity. The large white arrow represents the ballistic trajectory of an inertial detector. The small arrows represent the Killing vector field.

In the Minkowski and Rindler coordinate systems, the metric is

$$ds^2 = c^2\,dt^2 - dx^2 = L^2e^{2\xi}(d\theta^2 - d\xi^2). \tag{2}$$

This metric admits a Killing vector field which is everywhere regular, namely $K^{\theta} = 1$, $K^{\xi} = 0$ (or, in the original coordinates, $cK^t = x$, $K^x = ct$). This vector field is time-like in the Rindler wedge $x > c|t|$ and, in particular, within the spacetime domain swept by the cavity. Therefore, it is legitimate to use θ as a time coordinate within the cavity ($\xi_1 < \xi < \xi_2$); the metric properties of the latter then appear *static*. An increase in cavity time, $\theta \to \theta' = \theta + \rho$, corresponds in the Minkowski coordinates to a Lorentz transformation: $(x \pm ct) \to (x' \pm ct') = e^{\pm\rho}(x \pm ct)$. The increment ρ is called the *rapidity* of the transformation, and is related to the relative velocity of the two frames by $v = c\tanh\rho$.

The existence of a time-like Killing vector implies the conservation of a physical quantity which is analogous to an energy. In the present case, this conserved quantity is *not* the one that is called 'energy' in the Minkowski coordinate system, but rather corresponds to the generator of a Lorentz boost, as explained earlier. Its conservation guarantees that a quantum detector, at rest within the cavity and prepared in its lowest 'energy' state (with the new meaning of the word 'energy'), cannot be spontaneously excited to a higher state if the field too is in its ground state (the vacuum). Therefore no Unruh effect will be observed in such an experimental setup.

The same argument is clearly valid for any uniformly accelerated three-dimensional cavity of arbitrary shape. By 'uniformly accelerated' we mean that the round-trip time of a light signal between any two points located on the cavity boundary is independent of time. For example, in our one-dimensional cavity, the round-trip time $\xi_1 \to \xi_2 \to \xi_1$, measured by an observer located at ξ_1, is $2Le^{\xi_1}(\xi_2 - \xi_1)/c$ and is indeed independent of θ.

While a detector which is uniformly accelerated together with the cavity is *not* excited by the vacuum fluctuations, it is nevertheless possible to observe, at least in principle, the Unruh effect by *releasing* a detector in the cavity. The freely falling detector will then appear to follow a ballistic trajectory as shown in figure 1, and to be accelerated with respect to the normal modes (standing waves) of the field. Conversely, in the Lorentz frame where the detector is at rest, these modes appear to be accelerated. It is their autocorrelation, at the location of the detector, which determines the probability of excitation of the latter.

In this article, we shall illustrate this process by means of a simple theoretical model, involving a massless scalar field ϕ in two spacetime dimensions. Such a field satisfies a conformally invariant wave equation which is, in Rindler coordinates,

$$\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial^2\phi}{\partial\xi^2} = 0 \tag{3}$$

by virtue of equation (2). The normal modes satisfying $\phi(\xi_1,\theta) = \phi(\xi_2,\theta) = 0$ are

$$\phi_n(\xi,\theta) = \sqrt{2\kappa/\pi}\,\sin[n\kappa(\xi - \xi_1)]e^{\pm in\kappa\theta} \tag{4}$$

where

$$\kappa = \pi/(\xi_2 - \xi_1) = \pi/\log(g_1/g_2). \tag{5}$$

Here, g_1 and g_2 denote the proper accelerations of the 'floor' and 'ceiling' of the cavity, respectively. For a cavity of laboratory size, and with reasonable accelerations, g_1 and g_2 are almost equal, and therefore κ is a large number. (The physical meaning of κ is that of an inverse cavity length, measured in units $g/\pi c^2$.)

We shall later need the vacuum expectation value (Wightman function)

$$\begin{aligned} W(\xi'\theta',\xi''\theta'') &= \langle 0|\phi(\xi',\theta')\phi(\xi'',\theta'')|0\rangle \\ &= \sum_n (\hbar/\pi n)\sin[n\kappa(\xi' - \xi_1)]\sin[n\kappa(\xi'' - \xi_1)]e^{in\kappa(\theta''-\theta')}. \end{aligned} \tag{6}$$

This sum can be explicitly evaluated, but it is preferable to leave it in the present form, which is more convenient for the calculations of section 3.

2. Transitions between discrete levels

We now turn our attention to the detector. The latter has discrete energy levels, and its interaction with the quantized field ϕ may cause transitions between these levels. To determine the transition rate, one cannot use the familiar Fermi golden rule, since the latter is valid only when there is a *continuum* of final states, and is not applicable to transitions between *discrete* levels. For the discrete case, other prescriptions can be found in the literature, that correspond to various interaction models. However, we have not found any algorithm whose proof would be valid under the conditions stipulated in the present problem. We have therefore derived the necessary formula *ab initio* from first-order perturbation theory. Our approach has enough generality to make the results derived here applicable to a large variety of other physical situations.

Consider two weakly coupled quantum systems, such as an 'atom' a (our detector) having discrete energy levels, and a 'background' b, which may be a quantized field, or a thermal bath, or any other physical object whose interaction with the atom may cause quantum transitions. In the absence of coupling, the Hamiltonian is $H_0 = H_a + H_b$, where H_a is time-independent and has a discrete spectrum,

$$H_a|m\rangle = E_m|m\rangle. \tag{7}$$

On the other hand, no such assumptions are made for H_b, which may explicitly depend on time. The states of the background are described by an *arbitrary* orthonormal basis $|\alpha\rangle$, and those of the combined system by the tensor product of these two bases:

$$|m\alpha\rangle \equiv |m\rangle \otimes |\alpha\rangle. \tag{8}$$

The coupling between the two systems is assumed, for simplicity, to be a direct product $H_{\text{int}} = A \otimes B$, where A and B denote two operators, belonging to the atom and the background, respectively. The complete Schrödinger equation thus is

$$\mathrm{i}\hbar\, \mathrm{d}\psi/\mathrm{d}t = (H_a + H_b + A \otimes B)\psi \tag{9}$$

where ψ is a linear combination of the basis vectors (8). The $\otimes$ sign will henceforth be omitted, as there is no risk of confusion. Obviously, the operators A and B have to be written in equation (9) in their Schrödinger representation. We assume that, in that representation, A has no explicit time dependence (for example, it may be the dipole moment of an atom) but no such assumption is made for B. We shall later also need the Heisenberg representation of B, which is

$$B_{\mathrm{H}}(t) = U^{\dagger}(t, t_0) B(t) U(t, t_0) \tag{10}$$

where $U(t, t_0)$ is a unitary operator satisfying

$$\mathrm{i}\hbar\, \mathrm{d}U(t, t_0)/\mathrm{d}t = H_b U(t, t_0) \tag{11}$$

with the initial condition $U(t_0, t_0) = 1$.

We now expand

$$\psi(t) = U(t, t_0) \sum_{m\alpha} \mathrm{e}^{-\mathrm{i}E_m t/\hbar} c_{m\alpha}(t) |m\alpha\rangle \tag{12}$$

where

$$c_{m\alpha}(t) = e^{iE_m t/\hbar} \langle m\alpha | U^\dagger(t, t_0) | \psi(t) \rangle. \tag{13}$$

It is straightforward to show that the Schrödinger equation (9) is equivalent to

$$i\hbar \, dc_{m\alpha}(t)/dt = \sum_{n\beta} e^{i(E_m - E_n)t/\hbar} A_{mn} \langle \alpha | B_H(t) | \beta \rangle c_{n\beta} \tag{14}$$

where $A_{mn} = \langle m|A|n\rangle$.

Assume that the initial state of the combined system is given by equation (12) with $c_{i0}(t_0) = 1$, and all other $c_{m\alpha}(t_0) = 0$. Here, $|0\rangle$ denotes one of the basis states of the background subsystem. It may be the vacuum, or a thermal state, or any other state resulting from the physical preparation of that background. We are interested in the probability of finding the atom in a prescribed final state $|f\rangle$, at time t, irrespective of the final state of the background. That probability is

$$P(t, t_0) = \sum_\alpha |c_{f\alpha}(t)|^2 \tag{15}$$

and depends on both the initial time t_0 and the final time t, if the background system is not invariant under time translations (for example, the situation sketched in figure 1 is not invariant under a shift of t).

For the given initial conditions, and for any $|f\rangle$ orthogonal to the initial state of the atom, equation (14) becomes, in first-order perturbation theory,

$$i\hbar \, dc_{f\alpha}(t)/dt = e^{i\omega t} A_{fi} \langle \alpha | B_H(t) | 0 \rangle \tag{16}$$

where $\omega = (E_f - E_i)/\hbar$. It follows that

$$i\hbar c_{f\alpha}(t) = A_{fi} \int_{t_0}^{t} e^{i\omega t'} \langle \alpha | B_H(t') | 0 \rangle \, dt' \tag{17}$$

and therefore

$$\begin{aligned} P(t, t_0) &= |A_{fi}/\hbar|^2 \int_{t_0}^{t} \int_{t_0}^{t} e^{i\omega(t''-t')} \sum_\alpha \langle 0 | B_H(t') | \alpha \rangle \langle \alpha | B_H(t'') | 0 \rangle \, dt' \, dt'' \\ &= |A_{fi}/\hbar|^2 \int_{t_0}^{t} \int_{t_0}^{t} e^{i\omega(t''-t')} \langle 0 | B_H(t') B_H(t'') | 0 \rangle \, dt' \, dt''. \end{aligned} \tag{18}$$

This result is valid as long as $\sum_f P(t, t_0) \ll 1$. For longer times, first-order perturbation theory becomes inadequate.

In the elementary cases that are commonly discussed in the literature, $B_H(t)$ is assumed to be a *stationary* random function, so that the expression

$$W(t', t'') = \langle 0 | B_H(t') B_H(t'') | 0 \rangle \tag{19}$$

depends only on the time difference $t'' - t'$. Moreover, it is usually assumed that, if $|t'' - t'|$ is larger than a brief 'coherence time', then $|W(t', t'')| \ll W(t', t')$.

However, none of these physical assumptions is valid in the present work, and therefore more care is needed here.

Let us change the integration variables from t' and t'' to

$$\tau = t'' - t' \tag{20}$$

and

$$\sigma = (t'' + t')/2. \tag{21}$$

The Jacobian of this transformation is 1. We first perform the integration over τ, for fixed σ, and we obtain

$$P(t, t_0) = |A_{fi}/\hbar|^2 \int_{t_0}^{t} \left[\int_{-T}^{T} e^{i\omega\tau} W(\sigma - \tau/2, \sigma + \tau/2)\, d\tau \right] d\sigma \tag{22}$$

where $T \equiv \tau_{max} \equiv -\tau_{min}$ is a function of σ, t, and t_0, as shown in figure 2.

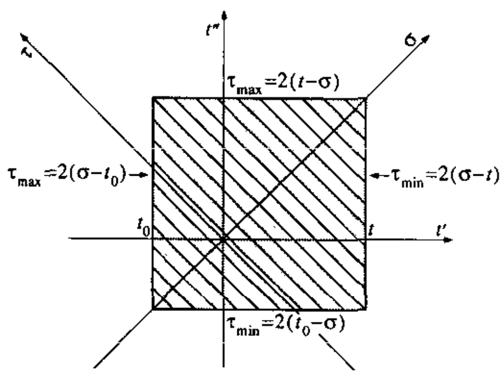

Figure 2. Integration limits of $\tau = t'' - t'$, for a fixed value of $\sigma = (t'' + t')/2$.

If the background system has a brief coherence time such that $W \to 0$ for large enough τ, the integration limits $\pm T$ can be replaced by $\pm\infty$, and then the integration over σ is trivial. The replacement of $\pm T$ by $\pm\infty$ may also be allowed even if W does *not* vanish for large τ, but rather has increasingly rapid oscillations which mutually cancel in the inner integral in equation (22). An example will be given in the next section. In such cases, it is possible to define an *instantaneous transition rate*

$$\Gamma(t) = dP(t, t_0)/dt = |A_{fi}/\hbar|^2 \int_{-\infty}^{\infty} e^{i\omega\tau} W(t - \tau/2, t + \tau/2)\, d\tau. \tag{23}$$

It may come as a surprise that this formula has the same general appearance as the one occurring in the fluctuation–dissipation theorem. For example, we obtain a similar result when we consider quantum transitions induced by an external, *classical, stochastic* force (Abragam 1961). On the other hand, there can be no doubt that the derivation given here fully takes into account the quantum nature of the background system. It does not rely on debatable semiclassical arguments (Senitzky 1978). It is based solely on the Schrödinger equation, which involves only reversible dynamics.

The dissipative nature of equation (23) is due to the sum over the final states of the background, in the definition of $P(t, t_0)$. When we discard the background degrees of freedom, there is an irreversible loss of information, and the density matrix of the atom (or the detector) turns from a pure state into a mixture.

3. Inertial detector in accelerated cavity

We shall assume for simplicity that our detector of vacuum fluctuations has only two internal energy levels, so that its internal state can be described with notation appropriate to a spin-$\frac{1}{2}$ particle. For a fully relativistic treatment of the detector's motion in spacetime, we shall assume that it is a point particle described by a local field

$$\chi(\boldsymbol{r}, t) = \sum_k (a_k |\uparrow\rangle + b_k |\downarrow\rangle) u_k(\boldsymbol{r}, t) + \text{HC} \tag{24}$$

where a_k and b_k are the annihilation operators for a detector in the normal mode $u_k(\boldsymbol{r}, t)$, with internal 'spin' up and down, respectively.

The Hamiltonian density for a detector moving in a scalar field ϕ will be taken as

$$\mathcal{H} = \mathcal{H}_\phi + \mathcal{H}_\chi + \chi^\dagger(\omega S_z + \lambda S_x \phi)\chi \tag{25}$$

where $\mathcal{H}_\chi$ involves only the detector's position variables; the internal variables S_x and S_z are the usual spin matrices (eigenvalues $\pm\hbar/2$); ω is a constant, such that $\hbar\omega$ is the energy separation of the two levels of the detector in the rest frame of the latter; and λ is the detector's coupling constant to the scalar field ϕ. The interaction term in this Hamiltonian density causes transitions between the internal levels of the detector. The occurrence of a transition may be interpreted as the detection of a vacuum fluctuation of the field.

If the energies involved in this process are low enough, compared to the detector's mass, there is no possibility of creation of detector–antidetector pairs, and it is a good approximation to ignore *virtual* pairs as well. This means that one can use a first-quantized formalism for the detector which is interacting with a second-quantized scalar field. In quantum theory, the state vector ψ becomes a function of the detector's position $\boldsymbol{r}$ and of its internal 'spin' variable, and a *functional* of the field ϕ (that is a function of the amplitudes of all the normal modes—an infinite number of variables). There is no inconsistency in this hybrid formalism. Its classical analogue is the interaction of a point particle with a continuous field, which is a standard approach to classical relativistic dynamics.

We further note that, in lowest-order perturbation theory, the detector's translational degrees of freedom are not affected by the scalar field ϕ. Therefore if the detector is massive enough to be localized in a wavepacket much smaller than the size of the cavity, its motion can be treated classically. This is a common approximation which is certainly valid for particle orbits in electron microscopes or in high-energy accelerators. The detector's position $\boldsymbol{r}$ will henceforth be considered as a prescribed function of the time t (not as an operator). In the Hamiltonian density (25), $\mathcal{H}_\chi$ becomes an irrelevant c-number that we may include in the background, or simply ignore.

We now return to the Schrödinger representation, in order to be able to use the results of the preceding section. Let Φ be the Schrödinger representation of the scalar field. With the approximations that were discussed above, the only remaining terms of the Hamiltonian are

$$H = H_\Phi + \omega S_z + \lambda S_x \Phi(\boldsymbol{r}, t). \tag{26}$$

In the last term, the field is taken at the detector's position r, which is a prescribed function of time.

To study a concrete example, we shall consider a segment of the detector's trajectory where x = constant (that is the detector is at rest in the xt Lorentz frame, as seen in figure 1). Let us evaluate the transition rate, $\Gamma(t)$, between the two levels of the detector. The symbols H_a and H_b that were used in the preceding section correspond to ωS_z and H_Φ, respectively. We expect $\Gamma(t)$ to depend on the detector's position, since there is no translation invariance in a cavity of finite size.

In equation (18), we set

$$|A_{fi}|^2 = (\lambda S_x)^2 = \hbar^2\lambda^2/4 \tag{27}$$

and we recall that the symbol ϕ that was used in section 1 referred to the Heisenberg representation of the free scalar field (namely, $\phi = \Phi_{\mathrm{H}}$). Therefore $W(t', t'')$ is the Wightman function given in equation (6), evaluated at a pair of points on the detector's trajectory. For our inertial detector (x = constant) we obviously have $|t| < x/c$, as seen in figure 1, but we shall moreover assume, in order to simplify the calculations, that $|t| \ll x/c$ (that is the position of the detector is always much closer to the apogee of its orbit within the cavity than to the extremities of the latter). We thus obtain, from equation (1),

$$\theta = \tanh^{-1}(ct/x) = (ct/x) + (ct/x)^3/3 + \cdots \tag{28}$$

and

$$\xi = \frac{1}{2}\log\frac{x^2 - c^2t^2}{L^2} = \log\frac{x}{L} - \frac{c^2t^2}{2x^2} - \cdots. \tag{29}$$

Again to simplify the calculations, it is convenient to choose x, the height of the orbit apogee, such that

$$\log(x/L) = (\xi_1 + \xi_2)/2. \tag{30}$$

This choice does not critically affect the final result, as will be shown later. With this value of x, we have

$$\sin[n\kappa(\xi - \xi_1)] = \sin(n\pi/2) - (c^2t^2/2x^2)n\kappa\cos(n\pi/2) + \mathrm{O}(t^4) \tag{31}$$

whence

$$\sin[n\kappa(\xi' - \xi_1)]\sin[n\kappa(\xi'' - \xi_1)] = \sin^2(n\pi/2) + \mathrm{O}(t^4). \tag{32}$$

If we had chosen a different value of x, there would be additional terms in (32) of the order of $\log(x/L) - (\xi_1 + \xi_2)/2$.

The absence of these terms and the omission of higher order corrections in (32) considerably simplify the sum in equation (6), which will now only run over odd n (we shall write it as $\sum'$):

$$W \simeq \frac{\hbar}{\pi}\sum_n{}' \frac{1}{n}\mathrm{e}^{\mathrm{i}n\kappa(\theta'' - \theta')}. \tag{33}$$

This expression is periodic in $\theta'' - \theta'$, with period $2\pi/\kappa$. Obviously, there is no finite 'coherence time' in our problem.

We now rewrite W in terms of the variables τ and σ, defined in the preceding section. From equation (28), we have

$$\theta'' - \theta' = (\tau uc/x) + (\tau^3 c^3/12x^3) + O(c^5 t^5/x^5) \tag{34}$$

where

$$u = 1 + (c\sigma/x)^2. \tag{35}$$

The inner integral in equation (22) becomes

$$I(\sigma, t, t_0) = \int_{-T}^{T} \sum_n{}' \frac{1}{n} \exp\left[i\tau\left(\omega + \frac{n\kappa uc}{x}\right) + i\tau^3 \frac{n\kappa c^3}{12x^3}\right] d\tau. \tag{36}$$

Each term of the sum can now be evaluated by the method of steepest descents (Erdélyi 1956). Let $a^2 = \omega + n\kappa uc/x$ and $b^2 = n\kappa c^3/4x^3$, and consider τ as a complex variable. The exponent in (36) has extrema (saddle points in the complex τ plane) at $\tau = \pm ia/b$. The integration path, originally defined as a segment on the real axis, can be distorted so as to pass through one of these saddle points. We write $\tau = \zeta + ia/b$, so that

$$\exp(ia^2\tau + ib^2\tau^3/3) = \exp(-ab\zeta^2 - 2a^3/3b + ib^2\zeta^3/3). \tag{37}$$

In the vicinity of the saddle point $\zeta = 0$, this expression behaves like a Gaussian with a peak of width $(ab)^{-1/2}$. This peak will give the main contribution to the integral if its width is well inside the original integration domain, that is if

$$T \gg [(\omega + n\kappa uc/x)(n\kappa c^3/4x^3)]^{-1/4} > (\omega\kappa c^3/4x^3)^{-1/4} \tag{38}$$

which means that both

$$(\sigma - t_0) \text{ and } (t - \sigma) \gg (x/c)(4c/x\omega\kappa)^{1/4}. \tag{39}$$

Recall that κ, which is given by equation (5), the inverse cavity length in units $g/\pi c^2$, and is a very large number for a cavity of reasonable size. Likewise, $x\omega/c$ is very large, if the detector has a reasonable energy splitting $\hbar\omega$. Therefore the inequalities (39) are well satisfied, except when $\sigma \simeq t_0$ and $\sigma \simeq t$, namely near the lower left and upper right corners of the square in figure 2.

We shall henceforth restrict our attention to observation times that are long enough,

$$t - t_0 \gg (x/c)(4c/x\omega\kappa)^{1/4} \tag{40}$$

so that both inequalities in (39) are satisfied, and yet are short enough to have $\sum_f P(t, t_0) \ll 1$, as otherwise perturbation theory would not be valid. These conditions on the observation time are similar to those that have to be imposed in the case of transitions to a continuum, if we want to obtain a roughly exponential decay law: if the observation time is too short or too long, the decay law of an unstable system is *not* exponential (Peres 1980). In the present case too, the existence of *three widely different time scales* is essential for the validity of our results.

If equation (40) is satisfied, we make only a negligible error if we replace, in equation (36), the finite integration limits $\pm T$ by $\pm\infty$. This is an example of the situation discussed at the end of the preceding section: the integrand in (36) does not vanish for large $|\tau|$, but it has increasingly rapid oscillations, due to the τ^3 term in the exponent, and these oscillations mutually cancel.

With the new integration limits $\pm\infty$, each term of the sum in (36) can be expressed by means of Airy functions (Abramowitz and Stegun 1968), giving

$$P(t,t_0) = \frac{\lambda^2\hbar}{2}\int_{t_0}^{t}\sum_n{}' \frac{1}{n}\left(\frac{4}{n\kappa}\right)^{1/3}\frac{x}{c}\,\mathrm{Ai}\left[\left(\frac{4}{n\kappa}\right)^{1/3}\frac{x}{c}\left(\omega+\frac{n\kappa uc}{x}\right)\right]\mathrm{d}\sigma. \tag{41}$$

Since $\omega \gg c/x$, the argument of the Airy function is large and we can replace that function by its asymptotic expansion

$$\mathrm{Ai}(z) \to \frac{1}{2\sqrt{\pi}}z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right). \tag{42}$$

This is a rapidly decreasing function of z, and the main contribution to the sum in equation (41) obviously comes from those n for which z is minimal. This minimum occurs for

$$2\kappa n \simeq x\omega/cu = N \tag{43}$$

which is a very large number. The fact that high-frequency modes are those which contribute most to $P(t,t_0)$ supports our intuitive guess that the precise location of the detector, namely $\xi_0 = \frac{1}{2}(\xi_1+\xi_2)$, has no critical influence and that the transition probability is a smooth function of x and t.

The next step is to expand the exponent in (42) into a Taylor series around its maximum, giving

$$\exp\left(-\tfrac{2}{3}z^{3/2}\right) \simeq \exp\left(-2\sqrt{3}Nu^{3/2}\right)\exp\left[(2n\kappa-N)^2u^{3/2}/\sqrt{3}N\right]. \tag{44}$$

As the main contribution to the sum in (41) comes from a relatively narrow range, $\Delta n \ll n \simeq N/2\kappa$, it is a good approximation to replace all the coefficients n by $N/2\kappa$ (except in the last exponent) and then to substitute

$$\sum_n{}' \ldots \to \int_0^\infty \ldots \mathrm{d}n/2. \tag{45}$$

Moreover, we can replace the lower integration limit 0 by $-\infty$, with a negligible error. This gives a Gaussian integral which is readily evaluated, and we finally obtain

$$P(t,t_0) = \frac{\lambda^2\hbar}{4\omega}\int_{t_0}^{t}\mathrm{e}^{-2\sqrt{3}u^{3/2}N}\,\mathrm{d}\sigma = \frac{\lambda^2\hbar}{4\omega}\int_{t_0}^{t}\mathrm{e}^{-2\sqrt{3u}x\omega/c}\,\mathrm{d}\sigma. \tag{46}$$

Therefore the transition rate is

$$\Gamma(t) = \frac{\mathrm{d}P(t,t_0)}{\mathrm{d}t} = \frac{\lambda^2\hbar}{4\omega}\mathrm{e}^{-2\sqrt{3u}x\omega/c}. \tag{47}$$

Note that $\sqrt{u}x = (x^2 + c^2t^2)^{1/2}$, so that Γ depends on the detector's position within the cavity, as we expected. It is maximal when the detector is at the highest point of its ballistic orbit. At that point, the relative acceleration of the vacuum fluctuations, with respect to the inertial detector, is $g = c^2/x$. We thus have

$$\Gamma = \frac{\lambda^2\hbar^2}{4E} e^{-2\sqrt{3}Ec/g\hbar} \tag{48}$$

where $E = \hbar\omega$ is the energy difference between the two levels.

This result holds only for transitions from the ground state to the excited state ($\omega > 0$), because the various approximations that we have just made would not be valid for $\omega < 0$. For de-excitations ($\omega < 0$), the Airy functions have a negative argument, which gives them an oscillatory behaviour (Abramowitz and Stegun 1968). The exponential factor in equation (48) must then be replaced by a complicated expression, of order unity, whose exact value is practically irrelevant: for reasonable values of E and g, the exponent in (48), is a large negative number, and Γ is so small that the probability of finding the detector in its ground state is always close to 1.

The transitions between the two levels will therefore reach equilibrium (if there is enough time for that, which depends on the strength of the coupling constant λ) when the respective occupation numbers are in a ratio close to $e^{-2\sqrt{3}Ec/g\hbar}$, i.e. as if the detector had a temperature $kT = g\hbar/2\sqrt{3}c$. This is higher than the Unruh temperature (for a linearly accelerated detector in a Minkowski vacuum) by a factor $\pi/\sqrt{3} \simeq 1.8$. However, this cannot be considered as a true thermal equilibrium, because the typical wavelength of thermal radiation at that temperature, about c^2/g, would be many orders of magnitude larger than the size of the cavity.

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